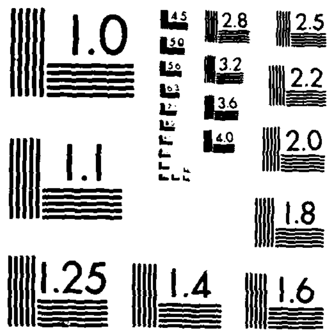
AD-A174 878 SOME RECENT RESULTS IN NONLINEAR FILTERING THEORY WITH FINITELY ADDITIVE (U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROC G KALLIANPUR NOV 85 1/1

UNCLASSIFIED TR-125 AFOSR-TR-86-2198 F49620-82-C-0009 F/G 12/1 NL


1.0
2.8
2.5
3.2
2.2
3.6
4.0
1.1
2.0
1.8
1.25
1.4
1.6

AFOSR-TR. 86-2198

AD-A174 878

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

SOME RECENT RESULTS IN NONLINEAR FILTERING

THEORY WITH FINITELY ADDITIVE WHITE NOISE

by

G. Kallianpur

Technical Report No. 125

November 1985

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY
NA

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE
NA

3. DISTRIBUTION/AVAILABILITY OF REPORT
Approved for Public Release; Distribution Unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
Technical Report No. 125

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR-TR- 86-2198

6a. NAME OF PERFORMING ORGANIZATION
University of North Carolina

6b. OFFICE SYMBOL *(If applicable)*

7a. NAME OF MONITORING ORGANIZATION
AFOSR/NM

6c. ADDRESS *(City, State and ZIP Code)*
Center for Stochastic Processes, Statistics Department, Philiips Hall 039-A, Chapel Hill, NC 27514

7b. ADDRESS *(City, State and ZIP Code)*
Bldg. 410
Bolling AFB, DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION
AFOSR

8b. OFFICE SYMBOL *(If applicable)*
nm

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
82C 0009
F49620 [redacted]

8c. ADDRESS *(City, State and ZIP Code)*
Bldg. 410
Bolling AFB, DC

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
6.1102F	2304	A5	

11. TITLE *(Include Security Classification)*
Some recent results in nonlinear filtering theory with finitely additive white noise

12. PERSONAL AUTHOR(S)
Kallianpur, G.

13a. TYPE OF REPORT
technical preprint

13b. TIME COVERED
FROM 9/85 TO 9/86

14. DATE OF REPORT *(Yr., Mo., Day)*
November 1985

15. PAGE COUNT
10

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB. GR.
XXXXX	XXXXXXXXX	XXXX

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
Keywords: xx

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Nonlinear filtering theory has been developed over the last few decades, largely, as an application of stochastic calculus. The theory (which will be referred to below as the conventional or stochastic calculus theory) has led to many important new advances in the subject and, indeed, given rise to problems of interest to stochastic calculus itself. When it comes to statistical applications, however, the approach based on stochastic calculus has many shortcomings which originate from the use of the Wiener process as a model for noise. This point has been recognized by many writers and has led to attempts to create a pathwise or robust version of the theory

In this article we present a very brief outline of an alternative approach developed recently in collaboration with R.L. Karandikar. In this theory, the Wiener process is replaced by finitely additive (f.a.) Gaussian which noise in the filtering model in which we also assume the independence of signal and noise.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL
~~Peggy Ravitch~~ Mary Crowley

22b. TELEPHONE NUMBER *(Include Area Code)*
~~919-962-2307~~ 767-5025

22c. OFFICE SYMBOL
AFOSR/NM

UNCLASSIFIED

SOME RECENT RESULTS IN NONLINEAR FILTERING THEORY WITH FINITELY ADDITIVE WHITE NOISE.

BY

G. Kallianpur
Center for Stochastic Processes
University of North Carolina, Chapel Hill, N.C., U.S.A.

1. Introduction.

Nonlinear filtering theory has been developed over the last few decades, largely, as an application of stochastic calculus. The theory (which will be referred to below as the conventional or stochastic calculus theory) has led to many important new advances in the subject and, indeed, given rise to problems of interest to stochastic calculus itself. When it comes to statistical applications, however, the approach based on stochastic calculus has many shortcomings which originate from the use of the Wiener process as a model for noise. This point has been recognized by many writers and has led to attempts to create a pathwise or robust version of the theory (For details, see the references in [1]).

In this article we present a very brief outline of an alternative approach developed recently in collaboration with R.L.Karandikar. In this theory, the Wiener process is replaced by finitely additive (f.a.) Gaussian white noise in the filtering model in which we also assume the independence of signal and noise.

Some of the new features of the white noise filtering theory are the following:

(1). No semimartingales or stochastic integrals need be used.

(2). A complete solution of the nonlinear filtering or prediction problem in the conventional theory involves, in general, solving a stochastic partial differential equation (SPDE). The latter is now replaced by "ordinary" partial differential equation in which the observation y of the white noise model enters as a parameter in the coefficients.

(3). Infinite dimensional (specifically, Hilbert-space valued) signal processes can be handled more naturally in the new set up and, in fact, the nonlinear filtering problem at this level of generality has been completely solved.

(4). The white noise theory yields results consistent with (and in most cases, under less restrictive conditions) the robust form of the conventional theory.

Research supported by AFOSR Contract No. F49620 82 C 0009.

Before describing our main results on filtering theory it is necessary to introduce the terminology and some of the basic definitions of the white noise calculus that enable us to obtain a suitable definition of conditional expectation. These definitions are more inclusive than the ones given in some of our previous publications. (All references relevant to this article will be found in [1] or [2]). A detailed development cannot be presented here for lack of space and will be available in [2].

2. Preliminaries on finitely additive white noise calculus.

Let (Ω,A) be a measurable space and let H be a real separable Hilbert space with inner product and norm denoted respectively by $(,)$ and $|\cdot|$. Denote by $\mathcal{P}(H)$, the family of orthogonal projectors on H with finite dimensional range. For $P \in \mathcal{P}(H)$ define $C_P = \{P^{-1}B,$ a Borel set $\subseteq$ Range $P\}$. Let $C = \bigcup_{P\in\mathcal{P}(H)} C_P$. Let $E = \Omega x H$ and define $\mathcal{E}$ to be the field $\bigcup_{P\in\mathcal{P}(H)} \mathcal{E}_P$ where $\mathcal{E}_P$ is the product σ-field AxC_P. $(E,\mathcal{E})$ is called a quasicylindrical measurable space. A quasicylindrical probability (QCP) β on $(E,\mathcal{E})$ is a finitely additive measure with $\beta(E) = 1$ and such that its restriction β_P to $\mathcal{E}_P$ is a (countably additive) probability measure. For our purposes, the most important example of a QCP is obtained as follows. Let Π be a complete probability measure on (Ω,A) and m, the canonical Gauss measure on (H,C), i.e., the f.a. measure with characteristic functional $\exp\{-\frac{1}{2}|h|^2\}$. The probability measures $\Pi\otimes m_P$ on $\mathcal{E}_P$ are consistent and determine on $(E,\mathcal{E})$ a unique QCP α such that $\alpha_P = \Pi\otimes m_P$. We write $\alpha = \Pi\otimes m$.

A function f defined on E and taking values in a Polish space S is called a cylinder function if $f(\omega,h) = f(\omega,(h,h_1),\ldots,(h,h_n))$ for some $h_1,\ldots,h_n$ in H and some measurable function $f_1: \Omega x \mathbb{R}^n \to S$. Let β be a QCP on $(E,\mathcal{E})$. With each cylinder function f is associated a lifting $R_\beta(f)$ which is a random variable (r.v.) on a "representation" propability space $(\tilde{\Omega},\tilde{A},\tilde{\Pi})$, the class of S-value d r.v.s. being denoted by $L(\tilde{\Omega},\tilde{A},\tilde{\Pi};S)$. The precise definition of a lifting requires the notation of the representation of a QCP β and will not be given here. (See [1]). We need a class, wider than the class of cylinder functions for which a lifting can be defined: $f \in L(E,\mathcal{E},\beta;S)$ if and only if for each $P \in \mathcal{P}(H)$ (= a set directed by $<$ where $P_1 < P_2$ if Range $P_1 \subseteq$ Range P_2), $f_P(\omega,h) := f(\omega,Ph)$ is $\mathcal{E}_P/\mathcal{B}(S)$-measurable and the net $\{R_\beta(f_P), P \in \mathcal{P}(H)\}$ converges in $\tilde{\Pi}$-probability to a limit which is defined to be the lifting $R_\beta(f)$ of f. It can be shown that $L(E,\mathcal{E},\beta:S)$ does not depend on the choice of representation of β.

Integration w.r.t β is defined as follows: Let

$$L_1(E,\mathcal{E},\beta) = \{f \in L(E,\mathcal{E},\beta;S): \int_{\tilde{\Omega}} |R_\beta(f)|d\tilde{\Pi} < \infty\} .$$

For $f \in L^1(E,\mathcal{E},\beta)$ define $\int_E f\, d\beta = \int_{\tilde{\Omega}} R_\beta(f)d\tilde{\Pi}$. The value of the integral does not depend on the choice of representation. We note that $(H,\mathcal{C})$ is a special example of a quasicylindrical measure space.

F.a. statistical model of filtering and estimation.

Definition. The identity mapping e on H as a mapping from $(H,\mathcal{C},m)$ to $(H,\mathcal{C})$ is called Gaussian white noise. Let $\xi: \Omega \to H$ be a $\mathcal{B}(H)/A$-measurable map. Writing $\xi(\omega,h) = \xi(\omega)$ and $e(\omega,h) = e(h)$, define $y: E \to H$ by

$$y = \xi + e \tag{1}$$

For $(\omega,h) \in E$, $y(\omega,h) = \xi(\omega) + e(h)$. (1) is the abstract model for filtering, y being the observation and ξ, the H-valued signal r.v.

Definition of a quasi cylindrical map (QCM) and conditional expectation:

Let $(E',\mathcal{E}')$ be a quasi-cylindrical measurable space and $\phi: E \to E'$ such that $\forall P' \in \mathcal{P}'$, $\exists P \in \mathcal{P}$ for which $\phi^{-1}(\mathcal{E}'_{P'}) \subseteq \mathcal{E}_P$. Then ϕ is called a QCM.

We mention some examples of QCM's.

(i) Let Q be an orthogonal projection on H and $H' = QH$.
Then $Q: H \to H'$ is a QCM.

(ii) y given by (1) is a QCM from $(E,\mathcal{E})$ to $(H,\mathcal{C})$.

(iii) Qy is a QCM from $(E,\mathcal{E},\alpha)$ to $(H',\mathcal{C}')$.

In our theory conditional expectations will be defined only with respect to QCM's. Let $f \in L'(E,\mathcal{E},\beta)$ and let ϕ be a QCM as defined above. Suppose the following conditions are satisfied:

(a) There exists $g \in L(E',\mathcal{E}',\beta', R)$, such that $go\phi \in L^1(E,\mathcal{E},\beta,R)$ and $R_{\beta'}(g) = R_\beta(go\phi)$;

(b) For all $F' \in \mathcal{E}'$, $\int_E f 1_{F'}(\phi)d\beta = \int_E go\phi 1_{F'}(\phi)d\beta$.

Then we define the conditional expectation

$$E_\beta(f|\phi) = go\phi .$$

The class of all g satisfying (a) and (b) will be denoted by $\mathcal{U}(\phi)$.

Remark 1. The existence of g and hence that of the conditional expectation is not asserted in the definition - an important departure from the situation in countably additive measure theory. Moreover, the conditional expectation, when it exists, is defined on the observation space itself and not on some representation space.

Bayes Formula and Zakai equation.

A finitely additive version of the Bayes formula is the principal tool in our theory.

Theorem 1. Let y be given by the abstract statistical model (1). Let Q be an arbitrary orthogonal projection on H, $H' = QH$. Let g be an integrable function on (Ω, A, Π). Then $E_\alpha(g|Qy)$ exists and is given by the formula

$$E_\alpha(g|Qy) = \frac{\sigma_Q(g,Qy)}{\sigma_Q(1,Qy)}$$

where for $h' \in H'$.

$$\sigma_Q(g,h') = \int_\Omega g(\omega) \exp\left[(h',Q\xi(\omega)) - \frac{1}{2}\left|Q\xi(\omega)\right|^2\right] d\Pi(\omega) .$$

We give an idea of the proof: Without loss of generality, assume $g \geq 0$ and $\int g d\Pi = 1$. Let

$$\Phi_g(C') = \int_E g(\omega) 1_{C'}(Qy(\omega,h)) d\alpha(\omega,h) \quad \text{when} \quad C' \in \mathcal{C}' .$$

Also let m' be the canonical Gauss measure on H', and

$$\mu'(B) = \int_\Omega g(\omega) 1_B(\xi(\omega)) d\Pi(\omega), \quad (B \text{ a Borel set in } H).$$

Then

$$\Phi_g(C') = \int_{H'} m'(C'-k) d\mu'(k) \quad \text{and} \quad \Phi_g \ll m'$$

with Radon-Nikodym derivative $\frac{d\Phi_g}{dm'}(h') = \sigma_Q(g,h')$. Hence from the definition of $n' = \alpha[Qy]^{-1}$ we have

$$\Phi_g(C') = \int_{H'} I_{C'}(h') \frac{\sigma_Q(g,h')}{\sigma_Q(1,h')} dn'(h') .$$

Both assertions of the theorem follow once we show that $\frac{\sigma_Q(g,\cdot)}{\sigma_Q(1,\cdot)} \in \mathcal{U}(Qy)$.

Applications to nonlinear filtering theory.

We now specialize the model (1) to the following:

(2) $$y_s = h_s(X_s) + e_s , \quad 0 \leq s \leq T , \quad \text{where}$$

(i) $X = (X_s, 0 \leq s \leq T)$ is a Markov process taking values in a Polish space S and defined on (Ω, A, P);

(ii) $h: [0,T] \times S \to R^m$ is a measurable map such that $\int_0^T |h_s(X_s)|^2 ds < \infty$ P-a.s ;

Let H be the real Hilbert space $\{\eta: [0,T] \to R^m =: |\eta| \in L^2[0.T]\}$, $\xi_s(\omega) = h_s(X_s(\omega))$, $0 \le s \le T$, i.e. $\xi(\omega) \in H$. The precise measurability conditions on X and h will be omitted;

(iii) $e = (e_s)$ is Gaussian white noise on H, independent of X.

In this model (y_s) is the observation process defined on the finitely additive probability space $(E, \mathcal{E}, P)$. Let Q_t be the orthogonal projection with range

$$H_t = \left\{\eta \in H: \int_t^T |\eta_s|^2 ds = 0\right\}.$$

The Bayes formula of Theorem 1 now takes the following form which is the starting point for deriving the necessary differential equations for the optimal filter:

For $f: S \to R$ such that $f o X_t \in L^1(\Omega, A, \Pi)$, $E_\alpha\left[f(X_t)|Q_t y\right] = \dfrac{\sigma_t(f,y)}{\sigma_t(1,y)}$ where $(Q_t y)(s) = y(s)$, for $0 \le s \le t$, $= 0$ for $t < s \le t$ and

$$\sigma_t(f,\eta) = \int_\Omega f(X_t(\omega)) \exp\left[\sum_{j=1}^m \int_0^t \eta_s^j h_s^j(X_s(\omega)) ds - \frac{1}{2} \sum_{j=1}^m (h_s^j(X_s(\omega))^2 ds\right] d\Pi(\omega).$$

Assume that X is an R^d-valued diffusion with extended generator

$$L_t = \frac{1}{2} \sum_{i,j} a_{ij}(t,x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_i b_i(t,x) \frac{\partial}{\partial x^i}.$$

We shall make the natural assumption that $C_0^{1,2}([0,\infty) \times R^d) \subseteq \mathcal{D}$. Then the "unnormalized conditional expectation" $\sigma_t(f,y)$ satisfies the following differential equation which is the finitely additive or white noise version of the Zakai equation of the conventional theory.

<u>Theorem 2</u>. Let $E\int_0^T |h_s(X_s)|^2 ds < \infty$. Then for $y \in H$, and $f \in \mathcal{D}(\{L_t\})$ (= domain of the generator L_t), $\frac{d}{dt}\sigma_t(f,y) = \sigma_t(\tilde{L}_t f, y) + \sum_{j=1} \sigma_t(h_t^j f, y) y_t^j$ where $\tilde{L}_t f = L_t f - \frac{1}{2}|h_t|^2 f$.

If $\sigma_t(f,\eta) = \int_{R^d} f(x) p_t(x,\eta) dx$, then $p_t(x,\eta)$ is called the unnormalized conditional density of the optimal filter. We now state one of the typical results pertaining to the filtering problem of a d-dimensional Markov signal process. It privides an alternative to solving the analogous stochastic partial differential equation (equivalently, an infinite dimensional SDE) for the conditional density.

<u>Theorem 3</u>. Assume the following conditions

(i) For all i,j, a_{ij} are bounded, and

(ii) $\sum_{i,j=1}^{d} a_{ij}(t,x)z_i z_j \geq K_1|z|^2$, $(K_1 > 0)$ for all (t,x) and $z = (z_1,\ldots,z_d) \in R^d$.

(iii) a_{ij}, $\frac{\partial a_{ij}}{\partial x^i}$, $\frac{\partial^2 a_{ij}}{\partial x^i \partial x^j}$, b_i, $\frac{\partial b_i}{\partial x^i}$ are locally Hölder continuous functions satisfying the growth condition $|g(t,x)| \leq K_2(1+|x|^2)^{\frac{1}{2}}$, $(0 < K_2 < \infty)$ for all (t,x).

(iv) h^k, $\frac{\partial h^k}{\partial x^i}$, $\frac{\partial^2 h^k}{\partial x^i \partial x^j}$ and $\frac{\partial h^k}{\partial t}$ are locally Hölder continuous in (t,x). (h^k are the components of the vector h).

(v) h^k, $\frac{\partial h^k}{\partial t}$, $\sum_j a_{ij}\frac{\partial h^k}{\partial x^j}$, $\sum_{i,j} a_{ij}\left(\frac{\partial^2 h^k}{\partial x^i \partial x^j} + \frac{\partial h^k}{\partial x^i}\cdot\frac{\partial h^k}{\partial x^j}\right)$ and $\sum_i b_i \frac{\partial h^k}{\partial x^i}$ satisfy the growth condition in (iii). Finally note the following assumptions about the initial random variable X_0:

(vi) $E\exp\alpha|X_0|^2 < \infty$ for some $\alpha > 0$;

(vii) The distribution of X_0 has a continuous density φ satisfying

$$|\varphi(x)| \leq \exp\left\{K_3(1+|x|^2)\right\}^{\frac{1}{2}-\varepsilon}$$ for some $\varepsilon > 0$, and a constant $K_3 > 0$.

Then

(a) For every $y \in H$, there is a unique $p_t(x,y)$ which satisfies

$$\frac{\partial p_t(x,y)}{\partial t} = L_t^* p_t(x,y) + \left(\sum_{j=1}^{d} h_t^j(x)y_t^j - \frac{1}{2}|h_t(x)|^2\right)p_t(x,y) \text{ for a.e.t.,} \tag{3}$$

with the initial condition $p_0(x,y) = \varphi(x)$ and

$$p_t(x,y)\cdot\exp\left[-\sum_{j=1}^{m} h_t^j(x)\int_0^t y_s^j ds\right] \in G$$

where G is the class of $C^{1,2}([0,T]\times R^d)$ functions satisfying the growth condition in (iii).

(b) For all $y \in H$, the unique solution $p_t(x,y)$ of (3) is the unnormalized conditional density of the filtering problem.

(c) The mapping $y \to p_t(x,y)$ is continuous in the sense that if $y_n \to y$ in H, then $p_.(\cdot,y_n) \to p_.(\cdot,y)$ uniformly on compact subsets of $[0,T]\times R^d$.

Remark 2. If only conditions (ii), (iii) and (vii) are assumed and h is assumed to be locally Hölder continuous, then Eqn (3) can be shown to have a unique solution in the class G, provided $y \in H_0$ where $H_0 = \{y \in H: y_t \text{ is Hölder continuous}\}$.

Remark 3. A special application of the previous remark is to the so-called cubic sensor: Take $d=2$, $m=1$ and $h_s(x) \equiv h(x) = x_1^3 + x_2^3$, $x=(x_1,x_2)$. Then h is locally Hölder continuous and hence for every $y \in H_0$, the Zakai equation (3) has a unique solution in G provided the drift and diffusion coefficients satisfy conditions (i) - (iii).

3. Hilbert space-valued Markov signal. Measure-valued optimal filter.

Suppose that in the statistical model $y_s = h_s(X_s) + e_s$, $0 \le s \le T$, h is a measurable function from $[0,T] \times S \to K$ where K is an infinite dimensional separable Hilbert space such that

$$\int_0^T |h_s(X_s)|_K^2 \, ds < \infty \quad \Pi\text{-a.s.},$$

and $e=(e_s)$ is Gaussian white noise on $H = L^2([0,T];K)$. This is a situation where, the state space of the signal process is essentially infinite dimensional. In this case, there can be no conditional density since there is no Lebesgue measure in Hilbert space and one can only deal with measure-valued equations. Measure-valued SDE's of Itô type for the optimal filter have been studied by Kunita and by Szpirglas. In the white noise context, measure-valued differential equations (in which the observation path y occurs as a parameter) have been obtained and the existence and uniqueness of solutions have been established by Kallianpur and Karandikar in a recent paper (See [2]). Here, for the sake of completeness we state two of the results. Let Q_t be the orthogonal projector on H with range $H_t = \{y \in H: \int_t^T |\eta(u)|_K^2 d\eta = 0\}$. If B is any Borel set in the Polish space S and $\eta \in H$ then define

$$\Gamma_t(\eta)(B) = \int_\Omega 1_B(X_t(\omega)) \exp\left\{(\eta, Q_t\xi(\omega)) - \frac{1}{2}\|Q_t\xi(\omega)\|^2\right\} d\Pi(\omega)$$

and

$$F_t(\eta)(B) = [\Gamma_t(\eta)(S)]^{-1} \, \Gamma_t(\eta)(B).$$

In the above expressions, the inner product in H is given by

$$(\eta_1,\eta_2) = \int^T (\eta_1(s),\eta_2(s))_K ds, \quad \|\eta\| = (\eta,\eta) \quad \text{and} \quad \xi: (\Omega,A,\Pi) \to H$$

is defined by

$$\xi_t(\omega) = h_t(X_t(\omega)), \quad 0 \le t \le T \quad \text{if} \quad \int_0^T |h_s(X_s)|_K^2 ds < \infty,$$

$= 0$ otherwise. Let $M(S)$ be the class of countably additive, finite Borel measures on S. Then for each $\eta \in H$, $\Gamma_t(\eta)$ and $F_t(\eta)$ belong to $M(S)$. They are, respectively, the unnormalized and the (normalized) conditional distribution of X_t.

Theorem 4. For $0 \leq t \leq T$,

$$\Gamma_t, \quad F_t \in \mathcal{L}^*(H,\mathcal{C},n;\mathcal{M}(S))$$

and

$$\Gamma_t(y), \quad F_t(y) \in \mathcal{L}^*(E,\mathcal{E},\alpha;\mathcal{M}(S)).$$

Here the star denotes the subclass of elements f in $\mathcal{L}(E,\mathcal{E},\alpha;\mathcal{M}(S))$ s.t. $R_\alpha(f \circ P_k) \to R_\alpha(f)$ in probability for all $P_k \overset{s}{\to} I$, $\{P_k\} \subseteq \mathcal{P}(H)$.

We shall now impose the further restriction $|h_s(x)|_K \leq q(s)\ \forall\ x \in S$ where q is a measurable function on S s.t. $\int_0^T q^2(s)ds < \infty$. Let $\{T_t\}$ be the one parameter semi-group associated with $\hat{X}_t = (t,X_t)$. Let L be the extended generator of $\{T_t\}$ with domain $\mathcal{D}$. The precise definitions of T_t and L will not be given here. We use the notation $\langle f,\mu\rangle$ for $\int f d\mu$ where $\mu \in \mathcal{M}(S)$.

Theorem 5. For all $y \in H$, $\{\Gamma_t(y)\}$ satisfies the equation

$$\begin{aligned} \langle f(t,\cdot),\Gamma_t(y)\rangle &= \langle f(0,\cdot),\Gamma_0(y)\rangle + \int_0^t \langle (Lf)(s,\cdot),\Gamma_s(y)\rangle ds \\ &+ \int_0^t \langle \{(h_s(\cdot),y_s)_K - \tfrac{1}{2}|h_s(\cdot)|_K^2\} f(s,\cdot),\Gamma_s(y)\rangle ds \end{aligned} \tag{4}$$

for all $f \in \mathcal{D}$.

Furthermore, $\Gamma_t(y)$ is the unique solution of (4) in the class of Borel measures $\{K_t\}$ on S such that for all Borel sets A,

(5) $K_t(A)$ is a bounded, measurable function of t, and $K_0(A) = E_\Pi 1_A(X_0)$.

A similar result holds for $F_t(y)$. The unique solution can be obtained by a sequence of approximations $\Gamma_t^n(y)$ which converges to $\Gamma_t(y)$ uniformly in t, in total variation norm in $\mathcal{M}(S)$.

Markov property of the optimal filter.

The optimal nonlinear filter in the white noise theory provides a concrete example of a continuous time Markov process defined on a f.a. probability space. From Theorem 4 we see that $\Gamma_t(y)$ is an $\mathcal{M}(S)$-valued family of r.v.'s on $(E,\mathcal{E},\alpha)$. The process $\Gamma_t(y)$ is a Markov process on $(E,\mathcal{E},\alpha)$ w.r.t. $\{Q_t y\}$ in the following sense:

(i) There exists $g \in \mathcal{U}(Q_t y)$ s.t. $\Gamma_t(y) = g(Q_t y)$ for all t.

(ii) For $0 \leq s < t_1, \ldots, t_k$ and for all $f \in C_b(\mathcal{M}(S)^k)$, $(k, s, t_1, \ldots, t_k$ arbitrary$)$,

$$E_\alpha\left[f(\Gamma_{t_1}(y), \ldots, \Gamma_{t_k}(y)) \middle| Q_s y\right] = g_1(\Gamma_s(y))$$

for a suitable function g_1.

We observe without going into details that the filtrations (F_t^Y) w.r.t. which the Markov property is defined in the conventional theory is now replaced by the family of QCM's $\{Q_t y\}$, $t \in R_+$.

4. Consistency of the white noise theory with the stochastic calculus approach.

Most of the robustness results have been obtained only for the case when the signal X is an R^d-valued diffusion. The cases when (a) (X_t) is the solution of a S.D.E. driven by a general process of independent increments and (b) X_t is a Markov process taking values in an infinite dimensional Hilbert space will not be considered here. Problem (a) has recently been studied by H.P.Hucke in his thesis. Problem (b) remains to be solved.

An easily verifiable "internal" robustness property is that $\Gamma_t(h_n, \cdot) \to \Gamma_t(h, \cdot)$ in total variation norm as $h_n \to h$ in H. Let $(\tilde{\Omega}, \tilde{A}, \tilde{\Pi}) = (\Omega, A, \Pi) \otimes (\Omega_0, A_0, \Pi_0)$, where the second factor is m-dimensional Wiener space. Suppose (Y_t) is the observation process of the conventional filtering model defined on $(\tilde{\Omega}, \tilde{A}, \tilde{\Pi})$ and let F_t^Y be the observation filtration. By choosing a lifting R_α of α to $(\tilde{\Omega}, \tilde{A}, \tilde{\Pi})$ we obtain the following result which reveals the underlying connection between the two theories:

Theorem 6. If $f: R^d \to R$ s.t. $E|f(X_t)| < \infty$, $0 \leq t \leq T$, then

$$R_\alpha\left[E_\alpha(f(X_t) \mid Q_t y)\right] = E_{\tilde{\Pi}}\left(f(X_t) \mid F_t^Y\right), \quad \text{a.s.}$$

Let $\Omega_0^* = \{\omega_0 \in \Omega : \omega_0(\cdot)$ is Hölder continuous$\}$. Our central result on robustness is the following

Theorem 7. Assume the conditions of Theorem 3. Then there exists a continuous function $\hat{p}_\cdot(\cdot, Y): \Omega^* \to C([0,T] \otimes R^d)$ which is a version of the unnormalized conditional density of X_t given F_t^Y with the property that $\hat{p}_t(x, Z) = p_t(x, z)$, $0 \leq t \leq T$, $x \in R^d$ where $Z_t = \int_0^t z_s ds$, $z \in H$.

Filtering models with non-white Gaussian noise.

We may examine the question of robustness of the white noise approach from the point of view that is more usual in the statistical theory. Roughly speaking, our results are robust in the sense that the optimal filter changes only slightly if the Gaussian noise is no longer white but has a covariance operator which is close to the identity. We have not been able to prove this statement in its greatest generality. To our knowledge this question is not completely settled even in the conventional theory. We have been able to establish this kind of robustness for a.f.a. non-white Gaussian process whose covariance Σ has a bounded inverse. For reasons of space we cannot go into this work, done in collaboration with H.P.Hucke and R.L.Karandikar, and we content ourselves with stating the Bayes formula for this model.

Theorem 8. Let the filtering model (1) be given on the quasicylindrical probability space $(E,\mathcal{E},\alpha_\Sigma)$ where $(E,\mathcal{E})$ is as before, $\alpha_\Sigma = \Pi \otimes m_\Sigma$ and m_Σ is the f.a. Gaussian measure on $(H,\mathcal{C})$ with zero mean and covariance operator Σ. It is assumed that the selfadjoint, positive operator Σ has a bounded inverse. Let g be a Π-integrable r.v. Then $E_{\alpha_\Sigma}(g|y)$ exists and

$$E_{\alpha_\Sigma}(g|y) = \frac{\sigma_\Sigma(g,y)}{\sigma_\Sigma(1,y)} \quad \text{where}$$

$$\sigma_\Sigma(g,h) = \int_\Omega g(\omega)\exp\left[(\Sigma^{-1}\xi(\omega),h) - \frac{1}{2}\left|\Sigma^{-1}\xi(\omega)\right|^2\right]d\Pi(\omega)\ , \quad h \in H\ .$$

A direct proof is somewhat lengthy. A simpler way is to reduce the problem to the white noise case already considered. This can be done by introducing a new inner product in H, $[h,k] = (\Sigma^{-1}h,k)$ on which m_Σ is a white noise measure.

I wish to thank the Royal Norwegian Research Council and Professor D.Tjøstheim for the invitation to visit the University of Bergen where this article was written.

References.

[1]. G.Kallianpur and R.L.Karandikar, Some recent developments in nonlinear filtering theory, Acta Applicandae Mathematicae (1983), 1, 399-434.

[2]. G.Kallianpur and R.L.Karandikar, White noise calculus and nonlinear filtering theory, Special Invited Paper to appear in Annals of Probability, (1985).

END

1-87

DTIC